Section 10-5 BASIC DIFFERENTIATION PROPERTIES

- Constant Function Rule
- Power Rule
- Constant Multiple Property
- Sum and Difference Properties
- Applications

In the preceding section, we defined the derivative of f at x as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

if the limit exists, and we used this definition and a four-step process to find the derivatives of several functions. Now we want to develop some rules of differentiation. These rules will enable us to find the derivative of a great many functions without using the four-step process.

Before starting on these rules, we list some symbols that are widely used to represent derivatives.

NOTATION The Derivative

If $y = f(x)$, then

$$f'(x) \qquad y' \qquad \frac{dy}{dx}$$

all represent the derivative of f at x.

Each of these symbols for the derivative has its particular advantage in certain situations. All of them will become familiar to you after a little experience.

Constant Function Rule

If $f(x) = C$ is a constant function, then the four-step process can be used to show that $f'(x) = 0$ (see Problem 45 in Exercise 10-4). Thus,

The derivative of any constant function is 0.

THEOREM 1 CONSTANT FUNCTION RULE

If $y = f(x) = C$, then

$$f'(x) = 0$$

Also, $y' = 0$ and $dy/dx = 0$.

Note: When we write $C' = 0$ or $\frac{d}{dx}C = 0$, we mean that $y' = \frac{dy}{dx} = 0$ when $y = C$.

INSIGHT

The graph of $f(x) = C$ is a horizontal line with slope 0 (see Fig. 1), so we would expect that $f'(x) = 0$.

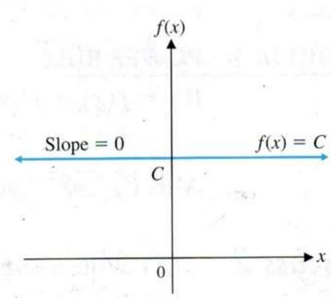

FIGURE 1

EXAMPLE 1 **Differentiating Constant Functions**

(A) If $f(x) = 3$, then $f'(x) = 0$. (B) If $y = -1.4$, then $y' = 0$.

(C) If $y = \pi$, then $\frac{dy}{dx} = 0$. (D) $\frac{d}{dx}23 = 0$

MATCHED PROBLEM 1 Find

(A) $f'(x)$ for $f(x) = -24$ (B) y' for $y = 12$

(C) $\frac{dy}{dx}$ for $y = -\sqrt{7}$ (D) $\frac{d}{dx}(-\pi)$

Power Rule

A function of the form $f(x) = x^k$, where k is a real number, is called a **power function.** The following elementary functions (see the inside front cover) are examples of power functions:

$$f(x) = x \qquad h(x) = x^2 \qquad m(x) = x^3$$
$$n(x) = \sqrt{x} \qquad p(x) = \sqrt[3]{x} \tag{1}$$

Explore & Discuss 1

(A) It is clear that the functions f, h, and m in (1) are power functions. Explain why the functions n and p are also power functions.

(B) The domain of a power function depends on the power. Discuss the domain of each of the following power functions:

$$r(x) = x^4 \qquad s(x) = x^{-4} \qquad t(x) = x^{1/4}$$
$$u(x) = x^{-1/4} \qquad v(x) = x^{1/5} \qquad w(x) = x^{-1/5}$$

The definition of the derivative and the four-step process introduced in the preceding section can be used to find the derivatives of many power functions. For example, it can be shown that

If $f(x) = x^2$, then $f'(x) = 2x$.
If $f(x) = x^3$, then $f'(x) = 3x^2$.
If $f(x) = x^4$, then $f'(x) = 4x^3$.
If $f(x) = x^5$, then $f'(x) = 5x^4$.

Notice the pattern in these derivatives. In each case, the power in f becomes the coefficient in f' and the power in f' is 1 less than the power in f. In general, for any positive integer n,

$$\text{If} \quad f(x) = x^n, \quad \text{then} \quad f'(x) = nx^{n-1}. \tag{2}$$

In fact, more advanced techniques can be used to show that (2) holds for *any* real number n. We will assume this general result for the remainder of the book.

THEOREM 2 POWER RULE

If $y = f(x) = x^n$, where n is a real number, then

$$f'(x) = nx^{n-1}$$

Also, $y' = nx^{n-1}$ and $dy/dx = nx^{n-1}$.

Explore & Discuss 2

(A) Write a verbal description of the power rule.

(B) If $f(x) = x$, what is $f'(x)$? Discuss how this derivative can be obtained from the power rule.

EXAMPLE 2 **Differentiating Power Functions**

(A) If $f(x) = x^5$, then $f'(x) = 5x^{5-1} = 5x^4$.

(B) If $y = x^{25}$, then $y' = 25x^{25-1} = 25x^{24}$.

(C) If $y = t^{-3}$, then $\frac{dy}{dt} = -3t^{-3-1} = -3t^{-4} = -\frac{3}{t^4}$.

(D) $\frac{d}{dx}x^{5/3} = \frac{5}{3}x^{(5/3)-1} = \frac{5}{3}x^{2/3}$.

MATCHED PROBLEM 2 Find

(A) $f'(x)$ for $f(x) = x^6$ (B) y' for $y = x^{30}$

(C) $\frac{dy}{dt}$ for $y = t^{-2}$ (D) $\frac{d}{dx}x^{3/2}$

In some cases, properties of exponents must be used to rewrite an expression before the power rule is applied.

EXAMPLE 3 **Differentiating Power Functions**

(A) If $f(x) = 1/x^4$, we can write $f(x) = x^{-4}$ and

$$f'(x) = -4x^{-4-1} = -4x^{-5}, \quad \text{or} \quad \frac{-4}{x^5}$$

(B) If $y = \sqrt{u}$, we can write $y = u^{1/2}$ and

$$y' = \frac{1}{2}u^{(1/2)-1} = \frac{1}{2}u^{-1/2}, \quad \text{or} \quad \frac{1}{2\sqrt{u}}$$

(C) $\frac{d}{dx}\frac{1}{\sqrt[3]{x}} = \frac{d}{dx}x^{-1/3} = -\frac{1}{3}x^{(-1/3)-1} = -\frac{1}{3}x^{-4/3}$, or $\frac{-1}{3\sqrt[3]{x^4}}$

MATCHED PROBLEM 3 Find

(A) $f'(x)$ for $f(x) = \frac{1}{x}$ (B) y' for $y = \sqrt[3]{u^2}$ (C) $\frac{d}{dx}\frac{1}{\sqrt{x}}$

Constant Multiple Property

Let $f(x) = ku(x)$, where k is a constant and u is differentiable at x. Then, using the four-step process, we have the following:

Step 1. $f(x + h) = ku(x + h)$

Step 2. $f(x + h) - f(x) = ku(x + h) - ku(x) = k[u(x + h) - u(x)]$

Step 3. $\frac{f(x + h) - f(x)}{h} = \frac{k[u(x + h) - u(x)]}{h} = k\left[\frac{u(x + h) - u(x)}{h}\right]$

Step 4. $f'(x) = \lim_{h\to 0}\frac{f(x + h) - f(x)}{h}$

$= \lim_{h\to 0} k\left[\frac{u(x + h) - u(x)}{h}\right]$ $\quad\lim_{x\to c} kg(x) = k\lim_{x\to c} g(x)$

$= k\lim_{h\to 0}\left[\frac{u(x + h) - u(x)}{h}\right]$ $\quad$Definition of $u'(x)$

$= ku'(x)$

Thus,

The derivative of a constant times a differentiable function is the constant times the derivative of the function.

THEOREM 3 CONSTANT MULTIPLE PROPERTY

If $y = f(x) = ku(x)$, then

$$f'(x) = ku'(x)$$

Also,

$$y' = ku' \qquad \frac{dy}{dx} = k\frac{du}{dx}$$

EXAMPLE 4 Differentiating a Constant Times a Function

(A) If $f(x) = 3x^2$, then $f'(x) = 3 \cdot 2x^{2-1} = 6x$.

(B) If $y = \frac{t^3}{6} = \frac{1}{6}t^3$, then $\frac{dy}{dt} = \frac{1}{6} \cdot 3t^{3-1} = \frac{1}{2}t^2$.

(C) If $y = \frac{1}{2x^4} = \frac{1}{2}x^{-4}$, then $y' = \frac{1}{2}(-4x^{-4-1}) = -2x^{-5}$, or $\frac{-2}{x^5}$.

(D) $\frac{d}{dx}\frac{0.4}{\sqrt{x^3}} = \frac{d}{dx}\frac{0.4}{x^{3/2}} = \frac{d}{dx}0.4x^{-3/2} = 0.4\left[-\frac{3}{2}x^{(-3/2)-1}\right]$

$= -0.6x^{-5/2}$, or $-\frac{0.6}{\sqrt{x^5}}$

MATCHED PROBLEM 4 Find

(A) $f'(x)$ for $f(x) = 4x^5$ (B) $\frac{dy}{dt}$ for $y = \frac{t^4}{12}$

(C) y' for $y = \frac{1}{3x^3}$ (D) $\frac{d}{dx}\frac{0.9}{\sqrt[3]{x}}$

Sum and Difference Properties

Let $f(x) = u(x) + v(x)$, where $u'(x)$ and $v'(x)$ exist. Then, using the four-step process, we have the following:

Step 1. $f(x + h) = u(x + h) + v(x + h)$

Step 2. $f(x + h) - f(x) = u(x + h) + v(x + h) - [u(x) + v(x)]$

$$= u(x + h) - u(x) + v(x + h) - v(x)$$

Step 3. $\frac{f(x + h) - f(x)}{h} = \frac{u(x + h) - u(x) + v(x + h) - v(x)}{h}$

$$= \frac{u(x + h) - u(x)}{h} + \frac{v(x + h) - v(x)}{h}$$

Step 4. $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$

$$= \lim_{h \to 0}\left[\frac{u(x + h) - u(x)}{h} + \frac{v(x + h) - v(x)}{h}\right]$$

$\lim_{x \to c}[g(x) + h(x)] = \lim_{x \to c} g(x) + \lim_{x \to c} h(x)$

$$= \lim_{h \to 0}\frac{u(x + h) - u(x)}{h} + \lim_{h \to 0}\frac{v(x + h) - v(x)}{h}$$

$$= u'(x) + v'(x)$$

Thus,

The derivative of the sum of two differentiable functions is the sum of the derivatives of the functions.

Similarly, we can show that

The derivative of the difference of two differentiable functions is the difference of the derivatives of the functions.

Together, we then have the **sum and difference property** for differentiation:

THEOREM 4 SUM AND DIFFERENCE PROPERTY

If $y = f(x) = u(x) \pm v(x)$, then

$$f'(x) = u'(x) \pm v'(x)$$

Also,

$$y' = u' \pm v' \qquad \frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$$

Note: This rule generalizes to the sum and difference of any given number of functions.

With the preceding rule and the other rules stated previously, we will be able to compute the derivatives of all polynomials and a variety of other functions.

EXAMPLE 5 Differentiating Sums and Differences

(A) If $f(x) = 3x^2 + 2x$, then

$$f'(x) = (3x^2)' + (2x)' = 3(2x) + 2(1) = 6x + 2$$

(B) If $y = 4 + 2x^3 - 3x^{-1}$, then

$$y' = (4)' + (2x^3)' - (3x^{-1})' = 0 + 2(3x^2) - 3(-1)x^{-2} = 6x^2 + 3x^{-2}$$

(C) If $y = \sqrt[3]{w} - 3w$, then

$$\frac{dy}{dw} = \frac{d}{dw}w^{1/3} - \frac{d}{dw}3w = \frac{1}{3}w^{-2/3} - 3 = \frac{1}{3w^{2/3}} - 3$$

(D) $$\frac{d}{dx}\left(\frac{5}{3x^2} - \frac{2}{x^4} + \frac{x^3}{9}\right) = \frac{d}{dx}\frac{5}{3}x^{-2} - \frac{d}{dx}2x^{-4} + \frac{d}{dx}\frac{1}{9}x^3$$

$$= \frac{5}{3}(-2)x^{-3} - 2(-4)x^{-5} + \frac{1}{9}\cdot 3x^2$$

$$= -\frac{10}{3x^3} + \frac{8}{x^5} + \frac{1}{3}x^2$$

MATCHED PROBLEM 5 Find

(A) $f'(x)$ for $f(x) = 3x^4 - 2x^3 + x^2 - 5x + 7$

(B) y' for $y = 3 - 7x^{-2}$

(C) $\dfrac{dy}{dv}$ for $y = 5v^3 - \sqrt[4]{v}$

(D) $\dfrac{d}{dx}\left(-\dfrac{3}{4x} + \dfrac{4}{x^3} - \dfrac{x^4}{8}\right)$

APPLICATIONS

EXAMPLE 6 **Instantaneous Velocity** An object moves along the y axis (marked in feet) so that its position at time x (in seconds) is

$$f(x) = x^3 - 6x^2 + 9x$$

(A) Find the instantaneous velocity function v.

(B) Find the velocity at $x = 2$ and $x = 5$ seconds.

(C) Find the time(s) when the velocity is 0.

SOLUTION (A) $v = f'(x) = (x^3)' - (6x^2)' + (9x)' = 3x^2 - 12x + 9$

(B) $f'(2) = 3(2)^2 - 12(2) + 9 = -3$ feet per second

$f'(5) = 3(5)^2 - 12(5) + 9 = 24$ feet per second

(C) $v = f'(x) = 3x^2 - 12x + 9 = 0$ Factor 3 out of each term.

$3(x^2 - 4x + 3) = 0$ Factor the quadratic term.

$3(x - 1)(x - 3) = 0$ Use the zero properly.

$x = 1, 3$

Thus, $v = 0$ at $x = 1$ and $x = 3$ seconds.

MATCHED PROBLEM 6 Repeat Example 6 for $f(x) = x^3 - 15x^2 + 72x$.

EXAMPLE 7 **Tangents** Let $f(x) = x^4 - 6x^2 + 10$.

(A) Find $f'(x)$.

(B) Find the equation of the tangent line at $x = 1$.

(C) Find the values of x where the tangent line is horizontal.

SOLUTION (A) $f'(x) = (x^4)' - (6x^2)' + (10)'$

$= 4x^3 - 12x$

(B) $y - y_1 = m(x - x_1)$ $y_1 = f(x_1) = f(1) = (1)^4 - 6(1)^2 + 10 = 5$

$y - 5 = -8(x - 1)$ $m = f'(x_1) = f'(1) = 4(1)^3 - 12(1) = -8$

$y = -8x + 13$ Tangent line at $x = 1$

(C) Since a horizontal line has 0 slope, we must solve $f'(x) = 0$ for x:

$f'(x) = 4x^3 - 12x = 0$ Factor $4x$ out of each term.

$4x(x^2 - 3) = 0$ Factor the difference of two squares.

$4x(x + \sqrt{3})(x - \sqrt{3}) = 0$ Use the zero properly.

$x = 0, -\sqrt{3}, \sqrt{3}$

MATCHED PROBLEM 7 Repeat Example 7 for $f(x) = x^4 - 8x^3 + 7$.

Answers to Matched Problems

1. All are 0.
2. (A) $6x^5$ (B) $30x^{29}$ (C) $-2t^{-3} = -2/t^3$ (D) $\frac{3}{2}x^{1/2}$
3. (A) $-x^{-2}$, or $-1/x^2$ (B) $\frac{2}{3}u^{-1/3}$, or $2/(3\sqrt[3]{u})$ (C) $-\frac{1}{2}x^{-3/2}$, or $-1/(2\sqrt{x^3})$
4. (A) $20x^4$ (B) $t^3/3$ (C) $-x^{-4}$, or $-1/x^4$ (D) $-0.3x^{-4/3}$, or $-0.3/\sqrt[3]{x^4}$
5. (A) $12x^3 - 6x^2 + 2x - 5$ (B) $14x^{-3}$, or $14/x^3$
   (C) $15v^2 - \frac{1}{4}v^{-3/4}$, or $15v^2 - 1/(4v^{3/4})$ (D) $3/(4x^2) - (12/x^4) - (x^3/2)$

6. (A) $v = 3x^2 - 30x + 72$
(B) $f'(2) = 24$ ft/s; $f'(5) = -3$ ft/s
(C) $x = 4$ and $x = 6$ seconds

7. (A) $f'(x) = 4x^3 - 24x^2$
(B) $y = -20x + 20$
(C) $x = 0$ and $x = 6$

Exercise 10-5

Find the indicated derivatives in Problems 1–18.

1. $f'(x)$ for $f(x) = 7$
2. $\frac{d}{dx}3$
3. $\frac{dy}{dx}$ for $y = x^9$
4. y' for $y = x^6$
5. $\frac{d}{dx}x^3$
6. $g'(x)$ for $g(x) = x^5$
7. y' for $y = x^{-4}$
8. $\frac{dy}{dx}$ for $y = x^{-8}$
9. $g'(x)$ for $g(x) = x^{8/3}$
10. $f'(x)$ for $f(x) = x^{9/2}$
11. $\frac{dy}{dx}$ for $y = \frac{1}{x^{10}}$
12. y' for $y = \frac{1}{x^{12}}$
13. $f'(x)$ for $f(x) = 5x^2$
14. $\frac{d}{dx}(-2x^3)$
15. y' for $y = 0.4x^7$
16. $f'(x)$ for $f(x) = 0.8x^4$
17. $\frac{d}{dx}\left(\frac{x^3}{18}\right)$
18. $\frac{dy}{dx}$ for $y = \frac{x^5}{25}$

Problems 19–24 refer to functions f and g that satisfy $f'(2) = 3$ and $g'(2) = -1$. In each problem, find $h'(2)$ for the indicated function h.

19. $h(x) = 4f(x)$
20. $h(x) = 5g(x)$
21. $h(x) = f(x) + g(x)$
22. $h(x) = g(x) - f(x)$
23. $h(x) = 2f(x) - 3g(x) + 7$
24. $h(x) = -4f(x) + 5g(x) - 9$

Find the indicated derivatives in Problems 25–48.

25. $\frac{d}{dx}(2x - 5)$
26. $\frac{d}{dx}(-4x + 9)$
27. $f'(t)$ if $f(t) = 2t^2 - 3t + 1$
28. $\frac{dy}{dt}$ if $y = 2 + 5t - 8t^3$
29. y' for $y = 5x^{-2} + 9x^{-1}$
30. $g'(x)$ if $g(x) = 5x^{-7} - 2x^{-4}$
31. $\frac{d}{du}(5u^{0.3} - 4u^{2.2})$
32. $\frac{d}{du}(2u^{4.5} - 3.1u + 13.2)$
33. $h'(t)$ if $h(t) = 2.1 + 0.5t - 1.1t^3$
34. $F'(t)$ if $F(t) = 0.2t^3 - 3.1t + 13.2$
35. y' if $y = \frac{2}{5x^4}$
36. w' if $w = \frac{7}{5u^2}$
37. $\frac{d}{dx}\left(\frac{3x^2}{2} - \frac{7}{5x^2}\right)$
38. $\frac{d}{dx}\left(\frac{5x^3}{4} - \frac{2}{5x^3}\right)$
39. $G'(w)$ if $G(w) = \frac{5}{9w^4} + 5\sqrt[3]{w}$
40. $H'(w)$ if $H(w) = \frac{5}{w^6} - 2\sqrt{w}$
41. $\frac{d}{du}(3u^{2/3} - 5u^{1/3})$
42. $\frac{d}{du}(8u^{3/4} + 4u^{-1/4})$
43. $h'(t)$ if $h(t) = \frac{3}{t^{3/5}} - \frac{6}{t^{1/2}}$
44. $F'(t)$ if $F(t) = \frac{5}{t^{1/5}} - \frac{8}{t^{3/2}}$
45. y' if $y = \frac{1}{\sqrt[3]{x}}$
46. w' if $w = \frac{10}{\sqrt[5]{u}}$
47. $\frac{d}{dx}\left(\frac{1.2}{\sqrt{x}} - 3.2x^{-2} + x\right)$
48. $\frac{d}{dx}\left(2.8x^{-3} - \frac{0.6}{\sqrt[3]{x^2}} + 7\right)$

For Problems 49–52, find

(A) $f'(x)$
(B) The slope of the graph of f at $x = 2$ and $x = 4$
(C) The equations of the tangent lines at $x = 2$ and $x = 4$
(D) The value(s) of x where the tangent line is horizontal

49. $f(x) = 6x - x^2$
50. $f(x) = 2x^2 + 8x$
51. $f(x) = 3x^4 - 6x^2 - 7$
52. $f(x) = x^4 - 32x^2 + 10$